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Padé approximants in the Wick–Cutkosky model

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Abstract. In the Wick–Cutkosky model the coupling constant is given as a Taylor series in powers of the centre of mass energy s. The series which is convergent in $|s|<4$ is continued by means of Padé approximants to the whole cut s-plane. The convergence of the Padé approximants is demonstrated, and agreement is found with results found before by different methods.

1. Introduction

The recent development in bound state models of hadrons has renewed the interest in solving the relativistic Bethe–Salpeter equation. Unfortunately, all physically interesting equations of this type are solvable only by highly complicated numerical procedures. Therefore it might be useful to investigate the models which can be solved analytically in order to gain insight into the more complicated models.

The outstanding model in this context is the Wick–Cutkosky model (Wick 1954, Cutkosky 1954), since in this model the interaction is motivated by field-theoretic arguments (exchange of a massless boson). For $s=0$ its solution can be written in closed form (Wick 1954), whereas for $s\neq 0$ a series expansion in s for the coupling constant λ has been given by zur Linden (1969) up to arbitrary order in s. A similar method has been used by Tanaka and Nakanishi (1975). This series is found to be convergent in $|s|<4$.

Its circle of convergence is limited by a branch point at $s=4$ with an associated cut extending from there to infinity. The occurrence of this cut may be understood as follows. Suppose we start with a coupling constant such that $s<4$. If we decrease λ the mass of the bound state will rise until s reaches the threshold for pair production at $s=4$. The mass of the bound state will move into the complex s-plane thus forming a resonance. If we insist on s staying real, this will only be possible for a complex λ. So we get a cut in the function $\lambda=\lambda(s)$. This will happen every time a new channel opens. So in the Wick–Cutkosky model, $\lambda(s)$ will have an infinite number of branches glued together along the cut extending from $s=4$ to infinity because at $s=4$ an infinite number of channels open due to mass zero of the exchanged bosons.

In field theory one would expect a corresponding left-hand cut starting at $s=0$ originated from the crossed channels. It will be absent in the Wick–Cutkosky model due to lack of crossing invariance. So the above mentioned cut is the only singularity in the complex s-plane which is expected on physical grounds.

Consequently, zur Linden (1971) demonstrated that one is able to find an expansion for $\lambda(s)$ convergent in the whole cut s-plane by a conformal mapping of the cut s-plane onto the unit circle. The cut is thereby mapped on the boundary of the circle.

In this paper, we propose a different method for the continuation of the solution. Starting with the series expansion for $\lambda(s)$ in powers of s, we construct the Padé approximants from the Taylor coefficients. For a review of Padé approximants see e.g. Baker (1975). Since we expect $\lambda(s)$ to be meromorphic outside the cut we can hope for convergence of the Padé approximants in the whole cut s-plane. The numerical calculations show that this is indeed true. Our results are in complete agreement with those found by zur Linden (1971).

The use of Padé approximants in strong interaction physics has been quite numerous in recent years (for a review see e.g. Basdevant 1973). But all these applications are different in spirit from the work presented here. Usually, the perturbation series for the T matrix is considered, and the Padé approximants are formed from their coefficients. The position of bound states is then inferred from the zeros of the denominator. We, however, study the coupling constant as a function of the bound state mass. This is possible because of the simple structure of the Wick–Cutkosky model.

In the next section we give the recurrence relations for the Taylor coefficients of $\lambda(s)$. The Padé approximants are constructed from it in § 3, and the results are exhibited.

2. Series expansion of $\lambda(s)$

In order to keep the paper sufficiently self-contained, we give in this section the recurrence relation for the Taylor coefficients of $\lambda(s)$. The derivation can be found in zur Linden (1969). The Wick–Cutkosky model is defined by the Bethe–Salpeter equation

$$[(p+\mathrm{i}\eta)^2+1][(p-\mathrm{i}\eta)^2+1]\psi(p)=\frac{\lambda}{\pi^2}\int\frac{\mathrm{d}^4k}{(p-k)^2}\psi(k). \tag{2.1}$$

The Wick rotation to four-dimensional Euclidean space has already been performed in (2.1). η is related to the centre of mass energy s by $s=4\eta^2$, and we have assumed the masses of the constituents to be equal to 1 thus fixing the momentum scale. The unequal mass case may be reduced to this case (Cutkosky 1954). We investigate the coupling constant λ as a function of the bound state mass η^2. The coefficients in the expansion

$$\lambda(\eta^2)=\sum_{i=0}^{\infty}\lambda_i\eta^{2i} \tag{2.2}$$

may be calculated from the recurrence relation

$$\lambda_m=(n+2q+p+1)(n+2q+p+2)$$
$$\times(a_{qm}+D^-_{q+1}a_{q+1,m-1}+D_qa_{q,m-1}+D^+_{q-1}a_{q-1,m-1}) \tag{2.3}$$

with the definitions

$$a_{q0}=1;\qquad a_{qj}=0\quad(j\neq0);\qquad a_{j0}=0\quad(j\neq q) \tag{2.4}$$

$$a_{rm} = \left(\sum_{t=|q-r|}^{m-1} a_{rt}\lambda_{m-t} - (n+2r+p+1)(n+2r+p+2) \right.$$
$$\left. \times (D^-_{r+1}a_{r+1,m-1} + D_r a_{r,m-1} + D^+_{r-1}a_{r-1,m-1}) \right)$$
$$\times [(n+2r+p+1)(n+2r+p+2) - (n+2q+p+1)(n+2q+p+2)]^{-1}. \tag{2.5}$$

The coefficients $D^\pm_n$, D_n can be found in zur Linden (1969). The solutions are labelled by the quantum numbers n and $k = 2q+p$. The numerical effort in computing λ_m to arbitrary order is minimal. An analytic computation of λ_m by using the symbolic programming language REDUCE (Hearn 1973) has revealed no deeper insight into the structure of λ_m besides that of rapidly increasing complication.

Since we are going to form Padé approximants it is of some interest to know whether the series (2.2) is a Stieltjes series. A series

$$f(z) = \sum_{j=0}^{\infty} f_j(-z)^j \tag{2.6}$$

is a series of Stieltjes if and only if (Baker 1975)

$$D(0,n) > 0 \qquad D(1,n) > 0 \qquad n = 0, 1, \ldots \tag{2.7}$$

with the determinants $D(m, n)$ defined by

$$D(m,n) = \begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \vdots & \vdots & & \vdots \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n} \end{vmatrix}. \tag{2.8}$$

Comparing (2.6) with (2.2) we have to put

$$f_j = (-1)^j \lambda_j \qquad j = 0, 1, \ldots. \tag{2.9}$$

From the recurrence relation (2.3) we find

$$\lambda_j < 0 \qquad j = 1, 2, \ldots. \tag{2.10}$$

Unfortunately, we have found this only numerically and have no mathematical proof for it. But taking this for granted, we find

$$D(0,1) = \lambda_0\lambda_2 - \lambda_1^2 < 0 \tag{2.11}$$

and hence the series (2.2) is not of the Stieltjes type.

3. Padé approximation for $\lambda(s)$

With the coefficients λ_t calculated from the recurrence relations (2.3) and (2.5), we can construct the Taylor series for $\lambda(s)$ from (2.2). The series is convergent inside the unit circle $|\eta^2| = 1$ i.e. for $s < 4$. The function $\lambda(s)$ in different regions of the s-plane may be calculated from it by using the Padé approximation.

The $[N, M]$ Padé approximation $f^{[N,M]}(z)$ to a function $f(z)$ is the ratio of two polynomials $P_N(z)$, $Q_M(z)$ of degree N and M respectively, which has the same $N+M$ first derivatives as $f(z)$ at $z=0$:

$$f^{[N,M]}(z) = P_N(z)/Q_M(z) = f(z) + \mathrm{O}(z^{N+M+1}). \tag{3.1}$$

The various approximations may be calculated most efficiently by using Wynn's formula (see Baker 1975):

$$\begin{aligned}&(f^{[N+1,M]}(z) - f^{[N,M]}(z))^{-1} + (f^{[N-1,M]}(z) - f^{[N,M]}(z))^{-1}\\ &\qquad = (f^{[N,M+1]}(z) - f^{[N,M]}(z))^{-1} + (f^{[N,M-1]}(z) - f^{[N,M]}(z))^{-1}.\end{aligned} \tag{3.2}$$

The approximations $f^{[N,0]}(z)$ are given by the partial sums of the Taylor series. If we formally define $f^{[-1,N]}(z) = 0$, $f^{[N,-1]}(z) = \infty$ the successive approximations are calculated from (3.2) according to the following scheme:

$$\begin{tikzcd}
f^{[0,0]} \arrow[d] & f^{[0,1]} & f^{[0,2]} \\
f^{[1,0]} \arrow[d] \arrow[ur] & f^{[1,1]} \arrow[ur] & \\
f^{[2,0]} \arrow[ur] & &
\end{tikzcd}$$

The convergence may be tested numerically in different ways. We have chosen to investigate the 'diagonal' approximations $f^{[N,N]}(z)$ for increasing values of N.

Once the Taylor coefficients λ_i $(i = 1, 100)$ are known from (2.3) and (2.5), the Padé approximations up to $N = 50$, $M = 50$ are calculated in less than a second of computing time. As would be expected the approximation converges in the whole s-plane outside the cut. The rapidity of the convergence depends on the separation of the respective value s from the cut. Similarly, the convergence gets slower as $s \to -\infty$ since the asymptotic behaviour of $\lambda(s)$ is certainly not given by a simple power law which is the only behaviour to be reproduced by the Padé approximation. To illustrate the speed of convergence, it will suffice to say that we have a six figure accuracy already from the [6, 6]-approximation at $\eta^2 = -10$ which is far outside the convergence circle of the Taylor series.

The results for various quantum numbers n, k are shown in figure 1. In this figure we have changed the variables η^2, λ to ϵ, Λ by the transformation

$$\epsilon = [1-(1-\eta^2)^{1/2}]/[1+(1-\eta^2)^{1/2}] \tag{3.3}$$

$$\Lambda(\epsilon) = (1+\epsilon)^2 \lambda(\eta^2) \tag{3.4}$$

so that for $\eta^2 = 1$, $\epsilon = 1$ and for $\eta^2 = -\infty$, $\epsilon = -1$. These variables have been used by zur Linden (1971) for his conformal mapping, so it will be simpler to compare his results with ours. We find complete agreement.

The Regge trajectories $n = n(\eta^2)$ which are degenerate for different values of l are found by keeping λ fixed. The leading trajectory is shown in figure 2. Again we find no deviation from zur Linden's results. Unfortunately, we are not able to extend the trajectory beyond $\eta^2 = 1$ since the Padé approximation does not converge in that region. We have not shown the daughter trajectories since they are also in agreement with previous results.

Summarizing, we have demonstrated the possibility of solving the Wick–Cutkosky model by a Padé approximation. The results are identical to those obtained by conformal mapping of the s-plane. We believe that our method will be superior if the exchanged particle has non-zero mass since then the analytic structure of $\lambda(s)$ will be

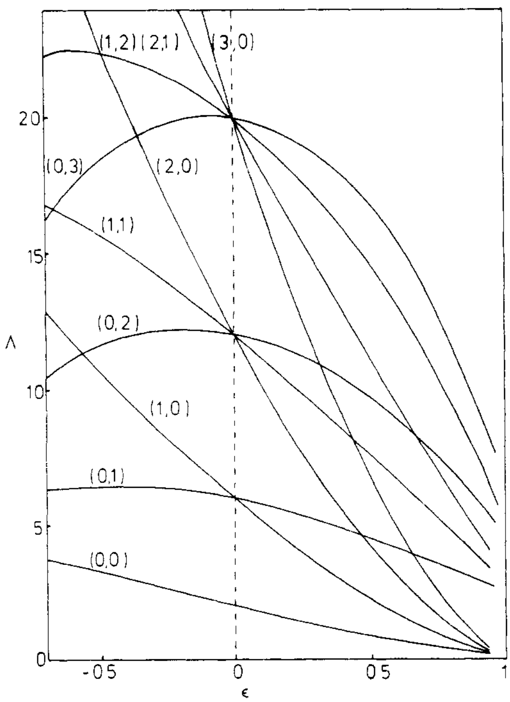

Figure 1. The coupling constant Λ as a function of ϵ for the lowest eigenvalues. The quantum numbers (n, k) are given in parentheses.

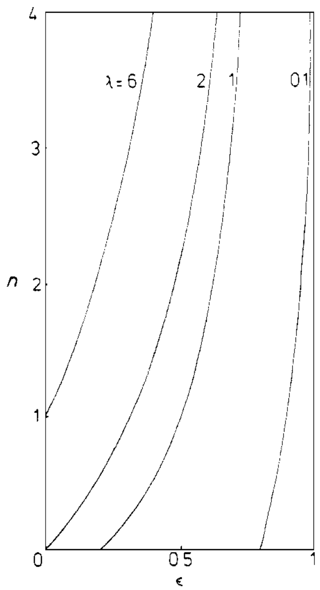

Figure 2. The leading trajectory $n_0(\epsilon)$ for several values of the coupling constant.

more involved (cuts starting at $s=(2+n\mu)^2$, $n=1, 2, \ldots$)). The best thing in this case might be the use of Padé approximations of the second kind (see Baker 1975) because in this more general case one cannot construct explicitly the Taylor series for $\lambda(s)$ around $s=0$. The investigation of this possibility will, however, be left to a future publication.

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